

REPRESENTATIONS OF THE q -DEFORMED ALGEBRA $U_q(\mathfrak{iso}_2)$

M. Havlíček

*Department of Mathematics, FNSPE, Czech Technical University
CZ-120 00, Prague 2, Czech Republic*

A. U. Klimyk

Institute for Theoretical Physics, Kiev 252143, Ukraine

S. Pošta

*Department of Mathematics, FNSPE, Czech Technical University
CZ-120 00, Prague 2, Czech Republic*

Abstract

An algebra homomorphism ψ from the q -deformed algebra $U_q(\mathfrak{iso}_2)$ with generating elements I, T_1, T_2 and defining relations $[I, T_2]_q = T_1$, $[T_1, I]_q = T_2$, $[T_2, T_1]_q = 0$ (where $[A, B]_q = q^{1/2}AB - q^{-1/2}BA$) to the extension $\hat{U}_q(\mathfrak{m}_2)$ of the Hopf algebra $U_q(\mathfrak{m}_2)$ is constructed. The algebra $U_q(\mathfrak{iso}_2)$ at $q = 1$ leads to the Lie algebra $\mathfrak{iso}_2 \sim \mathfrak{m}_2$ of the group $ISO(2)$ of motions of the Euclidean plane. The Hopf algebra $U_q(\mathfrak{m}_2)$ is treated as a Hopf q -deformation of the universal enveloping algebra of \mathfrak{iso}_2 and is well-known in the literature.

Not all irreducible representations of $U_q(\mathfrak{m}_2)$ can be extended to representations of the extension $\hat{U}_q(\mathfrak{m}_2)$. Composing the homomorphism ψ with irreducible representations of $\hat{U}_q(\mathfrak{m}_2)$ we obtain representations of $U_q(\mathfrak{iso}_2)$. Not all of these representations of $U_q(\mathfrak{iso}_2)$ are irreducible. The reducible representations of $U_q(\mathfrak{iso}_2)$ are decomposed into irreducible components. In this way we obtain all irreducible representations of $U_q(\mathfrak{iso}_2)$ when q is not a root of unity. A part of these representations turns into irreducible representations of the Lie algebra \mathfrak{iso}_2 when $q \rightarrow 1$. Representations of the other part have no classical analogue.

I. INTRODUCTION

Soon after definition of Drinfeld–Jimbo algebras $U_q(g)$, corresponding to semisimple Lie algebras g , the Hopf algebra $U_q(\mathfrak{m}_2)$ was defined [1] which is treated as a q -deformation of the universal enveloping algebra of the Lie algebra \mathfrak{iso}_2 of the group of motions of the Euclidean plane (the description of this group, its Lie algebra and their representations see, for example, in [2], Chap. 4).

However, there is another q -deformation of the universal enveloping algebra $U(\mathfrak{iso}_2)$ of the Lie algebra \mathfrak{iso}_2 which will be denoted by $U_q(\mathfrak{iso}_2)$. In the general form (that is, for $U(\mathfrak{iso}_n)$) such q -deformations were defined in [3]. The Hopf algebra $U_q(\mathfrak{m}_2)$ is related to the well-known quantum algebra $U_q(\mathfrak{sl}_2)$ while the associative algebra $U_q(\mathfrak{iso}_2)$ is connected with the nonstandard q -deformation $U_q(\mathfrak{so}_3)$ of the universal enveloping algebra $U(\mathfrak{so}_3)$ which is sometimes called the Fairlie algebra.

It is known that the theory of representations of the associative algebra $U_q(\mathfrak{so}_3)$ is richer than that of the algebra $U_q(\mathfrak{sl}_2)$ [4–6]. It was shown recently [7] that the theory of representations of the algebra $U_q(\mathfrak{iso}_2)$ is also richer than that of the algebra $U_q(\mathfrak{m}_2)$. In particular, the algebras $U_q(\mathfrak{so}_3)$ and $U_q(\mathfrak{iso}_2)$ have irreducible representations of nonclassical type (that is, representations which have no limit at $q \rightarrow 1$). The paper [7] is devoted to study of irreducible $*$ -representations of the algebra $U_q(\mathfrak{iso}_2)$ equipped with $*$ -structures. Irreducible representations of $U_q(\mathfrak{iso}_2)$ of the classical type are given in [8].

The aim of the present paper is to study irreducible representations of $U_q(\mathfrak{iso}_2)$ when this algebra is not equipped with some $*$ -structure and to clarify why irreducible representations of $U_q(\mathfrak{iso}_2)$ of the nonclassical type appear. We do this in the same way as in the case of representations of the algebra $U_q(\mathfrak{so}_3)$ in [6]. Namely, we relate the algebra $U_q(\mathfrak{iso}_2)$ with the extension $\hat{U}_q(\mathfrak{m}_2)$ of the Hopf algebra $U_q(\mathfrak{m}_2)$. This allows us to obtain representations of $U_q(\mathfrak{iso}_2)$ from those of the extended algebra $\hat{U}_q(\mathfrak{m}_2)$. We prove that if q is not a root of unity, then irreducible representations obtained in this way exhaust, up to equivalence, all irreducible representations of $U_q(\mathfrak{iso}_2)$.

II. THE ALGEBRAS $U_q(\mathfrak{iso}_2)$ AND $\hat{U}_q(\mathfrak{m}_2)$

The algebra $U_q(\mathfrak{iso}_2)$ is obtained by a q -deformation of the standard commutation relations

$$[I, T_2] = T_1, \quad [T_1, I] = T_2, \quad [T_2, T_1] = 0$$

of the Lie algebra \mathfrak{iso}_2 . So, $U_q(\mathfrak{iso}_2)$ is defined [7, 8] as the complex associative algebra with unit element generated by the elements I, T_1, T_2 satisfying the defining relations

$$[I, T_2]_q := q^{1/2}IT_2 - q^{-1/2}T_2I = T_1, \tag{1}$$

$$[T_1, I]_q := q^{1/2}T_1I - q^{-1/2}IT_1 = T_2, \tag{2}$$

$$[T_2, T_1]_q := q^{1/2}T_2T_1 - q^{-1/2}T_1T_2 = 0. \tag{3}$$

Note that the elements T_2 and T_1 of the algebra $U_q(\text{iso}_2)$ do not commute (as it is a case in the algebra iso_2 ; these elements correspond to shifts along the axes of the plane). We say that they q -commute, that is, $q^{1/2}T_2T_1 - q^{-1/2}T_1T_2 = 0$. This means that they generate the associative algebra determining the quantum plane.

Unfortunately, a Hopf algebra structure is not known on $U_q(\text{iso}_2)$. However, it can be embedded into the Hopf algebra $U_q(\text{isl}_2)$ as a Hopf coideal. (The algebra $U_q(\text{isl}_2)$ is the q -deformation of the universal enveloping algebra $U(\text{isl}_2)$ of the Lie algebra isl_2 of the inhomogeneous Lie group $ISL(2)$).

The relations (1)–(3) lead to the Poincaré–Birkhoff–Witt theorem for the algebra $U_q(\text{iso}_2)$. This theorem can be formulated as:

Proposition 1. *The elements $T_1^j T_2^k I^l$, $j, k, l = 0, 1, 2, \dots$, form a basis of the linear space $U_q(\text{iso}_2)$.*

Indeed, by using the relations (1)–(3) any product of the elements I, T_2, T_1 can be reduced to a sum of the elements $T_1^j T_2^k I^l$ with complex coefficients. Using the diamond lemma [9] (or its special case from Subsect. 4.1.5 in [10]) it is proved that these elements are linear independent. This proves Proposition 1.

Note that by (1) the element T_1 is not independent: it is determined by the elements I and T_2 . Thus, the algebra $U_q(\text{iso}_2)$ is generated by I and T_2 , but now instead of quadratic relations (1)–(3) we must take the relations

$$I^2 T_2 - (q + q^{-1}) I T_2 I + T_2 I^2 = -T_2, \quad (4)$$

$$I T_2^2 - (q + q^{-1}) T_2 I T_2 + T_2^2 I = 0, \quad (5)$$

which are obtained if we substitute the expression (1) for T_1 into (2) and (3). The equation $q^{1/2} I T_2 - q^{-1/2} T_2 I = T_1$ and the relations (4) and (5) restore the relations (1)–(3).

Note that the relation (5) is a relation of Serre's type in the definition of quantum algebras by V. Drinfeld and M. Jimbo. The relation (4) differs from Serre's relation by appearance of non-vanishing right hand side.

It is known that the element $C = T_1^2 + T_2^2$ from the universal enveloping algebra $U(\text{iso}_2)$ belongs to the center of this algebra. The analogue of this element in $U_q(\text{iso}_2)$ is the element $C_q = \frac{1}{2}(T_1 T_1' + T_1' T_1) + \frac{1}{2}(q + q^{-1}) T_2^2$, where $T_1' = q^{-1/2} I T_2 - q^{1/2} T_2 I$ (see [8]), that is $[C_q, X] := C_q X - X C_q = 0$ for all $X \in U(\text{iso}_2)$. This element can be reduced according to Proposition 1 to the form

$$C_q = q^{-1} T_1^2 + q T_2^2 + q^{-3/2} (1 - q^2) T_1 T_2 I. \quad (6)$$

The algebra $U_q(\text{iso}_2)$ is closely related to (but not coincides with) the quantum algebra $U_q(\mathfrak{m}_2)$. The last algebra is generated by the elements q^H, q^{-H}, E, F satisfying the relations

$$q^H q^{-H} = q^{-H} q^H = 1, \quad q^H E q^{-H} = q E, \quad q^H F q^{-H} = q^{-1} F, \quad [E, F] := EF - FE = 0. \quad (7)$$

In order to relate the algebras $U_q(\text{iso}_2)$ and $U_q(\mathfrak{m}_2)$ we need to extend $U_q(\mathfrak{m}_2)$ by the elements $(q^k q^H + q^{-k} q^{-H})^{-1}$, $k \in \mathbf{Z}$, in the sense of [11]. This extension $\hat{U}_q(\mathfrak{m}_2)$ is defined as the associative algebra (with unit element) generated by the elements

$$q^H, \quad q^{-H}, \quad E, \quad F, \quad (q^k q^H + q^{-k} q^{-H})^{-1}, \quad k \in \mathbf{Z},$$

satisfying the defining relations (7) of the algebra $U_q(\mathfrak{m}_2)$ and the following natural relations:

$$(q^k q^H + q^{-k} q^{-H})^{-1} (q^k q^H + q^{-k} q^{-H}) = (q^k q^H + q^{-k} q^{-H}) (q^k q^H + q^{-k} q^{-H})^{-1} = 1, \quad (8)$$

$$q^{\pm H} (q^k q^H + q^{-k} q^{-H})^{-1} = (q^k q^H + q^{-k} q^{-H})^{-1} q^{\pm H}, \quad (9)$$

$$(q^k q^H + q^{-k} q^{-H})^{-1} E = E (q^{k+1} q^H + q^{-k-1} q^{-H})^{-1}, \quad (10)$$

$$(q^k q^H + q^{-k} q^{-H})^{-1} F = F (q^{k-1} q^H + q^{-k+1} q^{-H})^{-1}. \quad (11)$$

III. THE ALGEBRA HOMOMORPHISM $U_q(\text{iso}_2) \rightarrow \hat{U}_q(\mathfrak{m}_2)$

The aim of this section is to give (in an explicit form) the homomorphism of the algebra $U_q(\text{iso}_2)$ to $\hat{U}_q(\mathfrak{m}_2)$.

Proposition 2. *There exists a unique algebra homomorphism $\psi : U_q(\text{iso}_2) \rightarrow \hat{U}_q(\mathfrak{m}_2)$ such that*

$$\psi(I_1) = \frac{i}{q - q^{-1}} (q^H - q^{-H}), \quad (12)$$

$$\psi(I_2) = (E - F)(q^H + q^{-H})^{-1}, \quad (13)$$

$$\psi(I_3) = (iq^{H-1/2}E + iq^{-H-1/2}F)(q^H + q^{-H})^{-1}, \quad (14)$$

where $q^{H+a} := q^H q^a$ for $a \in \mathbf{C}$.

Proof. In order to prove this proposition we have to show that the defining relations

$$\begin{aligned} q^{1/2} \psi(I) \psi(T_2) - q^{-1/2} \psi(T_2) \psi(I) &= \psi(T_1), \\ q^{1/2} \psi(T_1) \psi(I) - q^{-1/2} \psi(I) \psi(T_1) &= \psi(T_2), \\ q^{1/2} \psi(T_2) \psi(T_1) - q^{-1/2} \psi(T_1) \psi(T_2) &= 0. \end{aligned} \quad (15)$$

of $U_q(\text{iso}_2)$ are satisfied. Let us prove the relation (15). (Other relations are proved similarly.) Substituting the expressions (12)–(14) for $\psi(I)$, $\psi(T_2)$, $\psi(T_1)$ into (15) we obtain (after multiplying both sides of equality by $(q^H + q^{-H})$ on the right) the relation

$$\begin{aligned} & q(E - F)E q^H (q q^H + q^{-1} q^{-H})^{-1} + q(E - F)F q^{-H} (q^{-1} q^H + q q^{-H})^{-1} - \\ & - q E^2 q^H (q q^H + q^{-1} q^{-H})^{-1} - q^{-1} F E q^{-H} (q q^H + q^{-1} q^{-H})^{-1} + \\ & + q^{-1} E F q^H (q^{-1} q^H + q q^{-H})^{-1} + q F^2 q^{-H} (q^{-1} q^H + q q^{-H})^{-1} = 0. \end{aligned}$$

The formula (15) is true if and only if this relation is correct. We multiply both its sides by $(qq^H + q^{-1}q^{-H})(q^{-1}q^H + qq^{-H})$ on the right and obtain the relation in the algebra $U_q(\mathfrak{m}_2)$ (that is, without the expressions $(q^k q^H + q^{-k} q^{-H})^{-1}$). This relation is easily verified by using the defining relations (7) of the algebra $U_q(\mathfrak{m}_2)$. Proposition is proved.

IV. DEFINITION OF REPRESENTATIONS OF $U_q(\mathfrak{m}_2)$ AND $U_q(\mathfrak{iso}_2)$

From this point we assume that q is not a root of unity. Let us define representations of the algebras $U_q(\mathfrak{m}_2)$ and $U_q(\mathfrak{iso}_2)$.

Definition. By a representation π of $U_q(\mathfrak{m}_2)$ (respectively $U_q(\mathfrak{iso}_2)$) we mean a homomorphism of $U_q(\mathfrak{sl}_2)$ (respectively $U_q(\mathfrak{iso}_2)$) into the algebra of linear operators (bounded or unbounded) on a Hilbert space \mathcal{H} , defined on an everywhere dense invariant subspace \mathcal{D} , such that the operator $\pi(q^H)$ (respectively the operator $\pi(I)$) can be diagonalized, has a discrete spectrum (with finite multiplicities of spectral points if π is irreducible) and its eigenvectors belong to \mathcal{D} . Two representations π and π' of $U_q(\mathfrak{m}_2)$ (of $U_q(\mathfrak{iso}_2)$) on spaces \mathcal{H} and \mathcal{H}' , respectively, are called (algebraically) equivalent if there exist everywhere dense invariant subspaces $V \subset \mathcal{H}$ and $V' \subset \mathcal{H}'$ and a one-to-one linear operator $A : V \rightarrow V'$ such that $A\pi(a)v = \pi'(a)Av$ for all $a \in U_q(\mathfrak{m}_2)$ (respectively, for all $a \in U_q(\mathfrak{iso}_2)$) and $v \in V$.

Remark. Note that the element $I \in U_q(\mathfrak{iso}_2)$ corresponds to the homogeneous part of the motion group $ISO(2)$. As in the classical case, it is natural to demand in the definition of representations of $U_q(\mathfrak{iso}_2)$ that the operator $\pi(I)$ has a discrete spectrum (with finite multiplicities of spectral points for irreducible representations π). Such representations correspond to Harish-Chandra modules of Lie algebras. Note that irreducible $*$ -representations of $U_q(\mathfrak{iso}_2)$ without a requirement that $\pi(I)$ has a discrete spectrum were studied in [7]. It was shown there that the classification of irreducible $*$ -representations by self-adjoint operators in this case is equivalent to the classification of arbitrary families of bounded self-adjoint operators. The classification of irreducible representations (not obligatory $*$ -representations) in this case turn into unsolved problem.

The algebra $U_q(\mathfrak{m}_2)$ has the following non-trivial irreducible representations:

- (a) one-dimensional representations π_σ , $\sigma \in \mathbf{C}$, $\sigma \neq 0$, determined by the formulas $\pi_\sigma(q^H) = \sigma$, $\pi_\sigma(E) = \pi_\sigma(F) = 0$;
- (b) infinite dimensional representations π_{rs} , $r, s \in \mathbf{C}$, $r, s \neq 0$, acting on the Hilbert space \mathcal{H} with a basis $|m\rangle$, $m \in \mathbf{Z}$, by the formulas

$$\pi_{rs}(q^H)|m\rangle = sq^m|m\rangle, \quad \pi_{rs}(E)|m\rangle = r|m+1\rangle, \quad \pi_{rs}(F)|m\rangle = r|m-1\rangle, \quad m \in \mathbf{Z}. \quad (16)$$

We take $\mathcal{D} = \text{lin} \{|m\rangle \mid m \in \mathbf{Z}\}$. A direct verification shows:

Proposition 3. *The representations π_{rs} and $\pi_{r's'}$ ($r, s, r', s' \in \mathbf{C} \setminus \{0\}$) are equivalent if and only if $r = \pm r'$ and $s' = q^n s$ for some $n \in \mathbf{Z}$.*

Repeating the reasonings of Sect. 5.2 from [10] we easily prove

Proposition 4. *Every irreducible representation of $U_q(\mathfrak{m}_2)$ is equivalent to one of the representations (16) or is one-dimensional.*

Note that for $q \rightarrow 1$ the representations $\pi_{r,s}$ of $U_q(\mathfrak{m}_2)$ turn into irreducible representations of the universal enveloping algebra $U(\mathfrak{m}_2)$, that is, all irreducible representations of $U_q(\mathfrak{m}_2)$ are deformations of the corresponding irreducible representations of $U(\mathfrak{m}_2)$.

We try to extend representations $\pi_{r,s}$ of $U_q(\mathfrak{m}_2)$ to representations of the extension $\hat{U}_q(\mathfrak{m}_2)$ by using the relation

$$\pi((q^k q^H + q^{-k} q^{-H})^{-1}) := (q^k \pi(q^H) + q^{-k} \pi(q^{-H}))^{-1}, \quad k \in \mathbf{Z}. \quad (17)$$

Clearly, only those irreducible representations $\pi_{r,s}$ of $U_q(\mathfrak{m}_2)$ can be extended to $\hat{U}_q(\mathfrak{m}_2)$ for which the operators $q^k \pi(q^H) + q^{-k} \pi(q^{-H})$ are invertible. From formulas (16) it is clear that these operators are always invertible for the representations $\pi_{r,s}$, $s \neq \pm i q^n$, $n \in \mathbf{Z}$. (For the representations $\pi_{r,s}$, $s = \pm i q^n$ for some $n \in \mathbf{Z}$, some of these operators are not invertible since they have zero eigenvalue.) Denoting the extended representations by the same symbols $\pi_{r,s}$, we can formulate the following statement:

Proposition 5. *The algebra $\hat{U}_q(\mathfrak{m}_2)$ has the infinite dimensional representations $\pi_{r,s}$, $r, s \in \mathbf{C} \setminus \{0\}$, $s \neq \pm i q^n$ for all $n \in \mathbf{Z}$, given by the relations (16) and (17). The representations $\pi_{r,s}$ and $\pi_{r',s'}$ ($r, s, r', s' \in \mathbf{C} \setminus \{0\}$, $s, s' \neq \pm i q^n$ for all $n \in \mathbf{Z}$) are equivalent if and only if $r = \pm r'$ and $s' = q^m s$ for some $m \in \mathbf{Z}$. Any irreducible representation of $\hat{U}_q(\mathfrak{sl}_2)$ is equivalent to the representation $\pi_{r,s}$ for some r, s or is a one-dimensional representation.*

V. IRREDUCIBLE REPRESENTATIONS OF $U_q(\mathfrak{iso}_2)$

If π is a representation of the algebra $\hat{U}_q(\mathfrak{m}_2)$ on a space \mathcal{H} , then the mapping $R : U_q(\mathfrak{iso}_2) \rightarrow \mathcal{H}$ defined as the composition $R = \pi \circ \psi$, where ψ is the homomorphism from Proposition 1, is a (not necessary irreducible) representation of $U_q(\mathfrak{iso}_2)$.

Let us consider the representations

$$R_{rs} = \pi_{rs} \circ \psi$$

of $U_q(\mathfrak{iso}_2)$, where π_{rs} are the irreducible representations of $\hat{U}_q(\mathfrak{m}_2)$ from Proposition 5. Using formulas (16) and (12)–(14) we find that

$$R_{rs}(I)|m\rangle = i \frac{sq^m - s^{-1}q^{-m}}{q - q^{-1}}|m\rangle, \quad (18)$$

$$R_{rs}(T_2)|m\rangle = \frac{r}{sq^m + s^{-1}q^{-m}}\{|m+1\rangle + |m-1\rangle\}, \quad (19)$$

$$R_{rs}(T_1)|m\rangle = \frac{iq^{1/2}r}{sq^m + s^{-1}q^{-m}}\{sq^m|m+1\rangle - q^{-m}s^{-1}|m-1\rangle\}. \quad (20)$$

We consider that these operators are defined on the invariant subspace $\mathcal{D} \subset \mathcal{H}$ which is the span of the basis vectors $|m\rangle$. Thus we proved the following

Proposition 6. *Let $r, s \in \mathbf{C} \setminus \{0\}$, $s \neq \pm iq^n$ for all $n \in \mathbf{Z}$. Then the formulas (18)–(20) form a representation R_{rs} of the algebra $U_q(\text{iso}_2)$.*

We also have

Proposition 7. *The representations R_{rs} of Proposition 6 are irreducible if $s \neq \pm iq^{m+1/2}$, $m \in \mathbf{Z}$.*

Proof. Let $\{a\} := (sq^a - a^{-1}q^{-a})/(q - q^{-1})$. To prove this proposition we first note that since q is not a root of unity and $s \neq \pm iq^{m+1/2}$, $m \in \mathbf{Z}$, the eigenvalues $i\{m\}$, $m = 0, \pm 1, \pm 2, \dots$, of the operator $R_{rs}(I)$ are pairwise different.

Let $V \subset \mathcal{D}$ be an invariant subspace of the representation R_{rs} . We need to show that $V = \mathcal{D}$. Let $v = \sum_{m_i} \alpha_i |m_i\rangle \in V$, where $|m_i\rangle$ are eigenvectors of $R_{rs}(I)$ which are basis vectors of \mathcal{H} . (Note that the sum is finite since $v \in \mathcal{D}$.) Let us prove that $|m_i\rangle \in V$. We prove this for the case when $\mathbf{v} = \alpha_1 |m_1\rangle + \alpha_2 |m_2\rangle$. (The case of more number of summands is proved similarly.) We have

$$v' := R_{rs}(I)v = i\alpha_1 \{m_1\} |m_1\rangle + i\alpha_2 \{m_2\} |m_2\rangle.$$

. Since $v, v' \in V$, one derive that

$$i\{m_1\}v - v' = i\alpha_2(\{m_1\} - \{m_2\})|m_2\rangle \in V.$$

Since $\{m_1\} \neq \{m_2\}$, then $|m_2\rangle \in V$ and hence $|m_1\rangle \in V$.

In order to prove that $V = \mathcal{D}$ we obtain from (18) and (19) that

$$\{R_{rs}(T_1) - isq^{m_2+1/2}R_{rs}(T_2)\}|m_2\rangle = irq^{1/2}|m_2-1\rangle,$$

$$\{R_{rs}(T_1) + is^{-1}q^{-m_2+1/2}R_{rs}(T_2)\}|m_2\rangle = irq^{1/2}|m_2+1\rangle. \quad (20)$$

It follows from these relations that V contains the vectors $|m_2-1\rangle, |m_2-2\rangle, \dots$ and the vectors $|m_2+1\rangle, |m_2+2\rangle, \dots$. This means that $V = \mathcal{D}$ and the representation R_{rs} is irreducible. Proposition is proved.

Note that the representations R_{rs} of Proposition 7 turn into irreducible representations of the universal enveloping algebra $U(\text{iso}_2)$ when $q \rightarrow 1$. For this reason, they are called representations of the classical type.

Using Proposition 5 it is easy to show that the representations R_{rs} and $R_{r's'}$ of Proposition 7 are equivalent if and only if $r' = \pm r$ and $s = q^m s$ for some $m \in \mathbf{Z}$.

Proposition 8. *Let $r \in \mathbf{C} \setminus \{0\}$ and $s = \varepsilon iq^{m+1/2}$, where $m \in \mathbf{Z}$ and $\varepsilon \in \{1, -1\}$. Then the representation R_{rs} is reducible.*

Proof. The eigenvalues of the operator $R_{rs}(I)$ are

$$-\varepsilon \frac{q^n + q^{-n}}{q - q^{-1}}, \quad n = \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{5}{2}, \dots,$$

that is, every spectral point has multiplicity 2. The pairs of vectors $|-m+j\rangle$ and $|-m-j-1\rangle$, $j = 0, 1, 2, \dots$, are of the same eigenvalue. Let us define two subspaces V_1 and V_{-1} by the formulas $V_{\tilde{\varepsilon}} := \text{lin} \{|j\rangle_{\tilde{\varepsilon}} | j = 0, 1, 2, \dots\}$, where

$$|j\rangle_{\tilde{\varepsilon}} := |-m+j\rangle + \tilde{\varepsilon} i (-1)^j |-m-j-1\rangle, \quad j = 0, 1, 2, \dots \quad (21)$$

A direct calculation show that for $\tilde{\varepsilon} = 1$ and for $\tilde{\varepsilon} = -1$ we have

$$R_{rs}(I)|j\rangle_{\tilde{\varepsilon}} = -\varepsilon \frac{q^{j+1/2} + q^{-j-1/2}}{q - q^{-1}} |j\rangle_{\tilde{\varepsilon}}, \quad j = 0, 1, 2, \dots, \quad (22)$$

$$R_{rs}(T_2)|0\rangle_{\tilde{\varepsilon}} = -\frac{r}{q^{1/2} - q^{-1/2}} (\tilde{\varepsilon}|0\rangle_{\tilde{\varepsilon}} + i|1\rangle_{\tilde{\varepsilon}}), \quad (23)$$

$$R_{rs}(T_2)|j\rangle_{\tilde{\varepsilon}} = -\varepsilon \frac{ir}{q^{j+1/2} - q^{-j-1/2}} (|j+1\rangle_{\tilde{\varepsilon}} + |j-1\rangle_{\tilde{\varepsilon}}), \quad j = 1, 2, 3, \dots, \quad (24)$$

$$R_{rs}(T_1)|0\rangle_{\tilde{\varepsilon}} = \frac{r}{q^{1/2} - q^{-1/2}} (\tilde{\varepsilon}|0\rangle_{\tilde{\varepsilon}} + iq|1\rangle_{\tilde{\varepsilon}}), \quad (25)$$

$$R_{rs}(T_1)|j\rangle_{\tilde{\varepsilon}} = \frac{ir}{q^{j+1/2} - q^{-j-1/2}} (q^{j+1}|j+1\rangle_{\tilde{\varepsilon}} + q^{-j}|j-1\rangle_{\tilde{\varepsilon}}), \quad j = 1, 2, 3, \dots \quad (26)$$

These formulas show that the subspaces V_1 and V_{-1} are invariant with respect to the representation R_{rs} , that is this representation is reducible. Proposition is proved.

Let us denote the restrictions of the representation R_{rs} from Proposition 7 to the invariant subspaces V_1 and V_{-1} by $R_r^{\varepsilon,1}$ and $R_r^{\varepsilon,-1}$, respectively. It is seen from the formulas (22)–(26) that the operators are independent of s and the index s is omitted. These formulas show that R_{rs} , $s = \varepsilon i q^{m+1/2}$, is the direct sum of the representations $R_r^{\varepsilon,1}$ and $R_r^{\varepsilon,-1}$.

Proposition 9. *The representations $R_r^{\varepsilon,\tilde{\varepsilon}}$ and $R_{r'}^{\varepsilon',\tilde{\varepsilon}'}$ are equivalent if $(r, \varepsilon, \tilde{\varepsilon}) = (-r', \varepsilon', -\tilde{\varepsilon}')$.*

Proposition is easily proved by using Proposition 5 and formulas (22)–(26).

Theorem 1. (a) *Let r and r' be nonzero complex numbers such that $\text{Re } r > 0$ and $\text{Re } r' > 0$, and let $\varepsilon, \tilde{\varepsilon}, \varepsilon', \tilde{\varepsilon}' \in \{1, -1\}$. If $(\varepsilon, \tilde{\varepsilon}, r) \neq (\varepsilon', \tilde{\varepsilon}', r')$, then the representations $R_r^{\varepsilon,\tilde{\varepsilon}}$ and $R_{r'}^{\varepsilon',\tilde{\varepsilon}'}$ are irreducible and nonequivalent.*

(b) *Let $r \in \mathbf{C} \setminus \{0\}$, $\varepsilon, \tilde{\varepsilon} \in \{1, -1\}$, and let $r', s' \in \mathbf{C} \setminus \{0\}$, $s' \neq \pm i q^{m+1/2}$ for all $m \in \mathbf{Z}$. Then the representations $R_r^{\varepsilon,\tilde{\varepsilon}}$ and $R_{r's'}^{\varepsilon,\tilde{\varepsilon}}$ are nonequivalent.*

Proof. The irreducibility is proved in the same way as in Proposition 7. In order to prove a nonequivalence we note that the spectrum of the operator $R(I)$ for any of the representations $R_r^{+,+}$, $R_r^{-,+}$, $\text{Re } r > 0$, does not coincide with that of any of the

representations $R_r^{+, -}$, $R_{r'}^{-, -}$, $\operatorname{Re} r > 0$. Therefore, any of the representations $R_r^{+, +}$, $R_{r'}^{-, +}$, $\operatorname{Re} r > 0$, cannot be equivalent to some of the representations $R_r^{+, -}$, $R_{r'}^{-, -}$, $\operatorname{Re} r > 0$.

The operators $R_r^{\epsilon_1, \epsilon_2}(T_2)$, $\epsilon_1, \epsilon_2 = +, -$, are trace class operators. Their traces are nonzero (there exists only one nonzero diagonal matrix element with respect to the basis $\{|m\rangle'\}$ or the basis $\{|m\rangle''\}$). Since for $\operatorname{Re} r > 0$ and $\operatorname{Re} r' > 0$, $r \neq r'$, we have $\operatorname{Tr} R_r^{+, +}(T_2) \neq \operatorname{Tr} R_{r'}^{-, +}(T_2)$, then any of the representations $R_r^{+, +}$, $\operatorname{Re} r > 0$, cannot be equivalent to some of the representations $R_{r'}^{-, +}$, $\operatorname{Re} r' > 0$. It is proved similarly that any of the representations $R_r^{+, -}$, $\operatorname{Re} r > 0$, cannot be equivalent to some of the representations $R_{r'}^{-, -}$, $\operatorname{Re} r' > 0$. This prove the assertion (a). The assertion (b) is proved similarly. Proposition is proved.

Representations of Theorem 1 have no classical limit since at $q \rightarrow 1$ the denominators in (22)–(26) turn into zero. For this reason, these representations are called representations of non-classical type. There are no analogues of such representations for the Lie algebra \mathfrak{iso}_2 .

Theorem 2. *Every irreducible representation of $U_q(\mathfrak{iso}_2)$ is equivalent to one of the representations $R_{r,s}$ and $R_r^{\varepsilon, \tilde{\varepsilon}}$ or is one-dimensional. This means that the representations $R_{r,s}$ and $R_r^{\varepsilon, \tilde{\varepsilon}}$, $r, s, r' \in \mathbf{C} \setminus \{0\}$, $\varepsilon, \tilde{\varepsilon} \in \{1, -1\}$ defined by the relations (18)–(20) and (22)–(26), respectively, exhaust (up to equivalence) all irreducible representations of $U_q(\mathfrak{iso}_2)$.*

Proof. Let R be an irreducible representation of $U_q(\mathfrak{iso}_2)$. Then it follows from the definition of representations that $R(I)$ has some eigenvector $|0\rangle$. Thus there exists $s \in \mathbf{C}$, $s \neq 0$, such that

$$R(I)|0\rangle = i[0]_{q,s}|0\rangle,$$

where $[m]_{q,s} := (sq^m - s^{-1}q^{-m})/(q - q^{-1})$. Since R is irreducible there exists a complex number C such that $R(C_q) = C$ (see (6)). We define recursively the vectors

$$|j+1\rangle := R(iT_1 - s^{-1}q^{-j+1/2}T_2)|j\rangle, \quad j = 0, 1, 2, \dots, \quad (27)$$

$$|j-1\rangle := R(iT_1 + sq^{j+1/2}T_2)|j\rangle, \quad j = 0, -1, -2, \dots. \quad (28)$$

Some of these vectors may be linear dependent or be equal to 0. It follows from (1)–(3) and (6) that

$$R(I)|j\rangle = i[j]_{q,s}|j\rangle, \quad j \in \mathbf{Z}, \quad (29)$$

$$R(iT_1 + sq^{j+3/2}T_2)|j+1\rangle = -Cq|j\rangle, \quad j = 0, 1, 2, \dots, \quad (30)$$

$$R(iT_1 - s^{-1}q^{-j+3/2}T_2)|j-1\rangle = -Cq|j\rangle, \quad j = 0, -1, -2, \dots. \quad (31)$$

As a sample, we prove the relation (30) for $j \geq 0$:

$$\begin{aligned} R(iT_1 + sq^{j+3/2}T_2)|j+1\rangle &= R(iT_1 + sq^{j+3/2}T_2)R(iT_1 - s^{-1}q^{-j+1/2}T_2)|j\rangle = \\ &= R(-T_1^2 + isq^{j+3/2}T_2T_1 - is^{-1}q^{-j+1/2}T_1T_2 - q^2T_2^2)|j\rangle = \end{aligned}$$

$$\begin{aligned}
&= qR(-q^{-1}T_1^2 - qT_2^2 + isq^{j-3/2}T_1T_2 - is^{-1}q^{-j-1/2}T_1T_2)|j\rangle = \\
&= qR(-q^{-1}T_1^2 - qT_2^2 - q^{-3/2}(1 - q^2)T_1T_2I)|j\rangle = -qR(C_q)|j\rangle = -qC|j\rangle.
\end{aligned}$$

We obtain from (27) and (30) that

$$R(T_2)|j\rangle = -(s^{-1}q^{-j+1/2} + sq^{j+1/2})^{-1}(|j+1\rangle + Cq|j-1\rangle), \quad (32)$$

$$iR(T_1)|j\rangle = \frac{sq^{j+1/2}}{s^{-1}q^{-j+1/2} + sq^{j+1/2}}|j+1\rangle + Cq \left(\frac{sq^{j+1/2}}{s^{-1}q^{-j+1/2} + sq^{j+1/2}} \right) |j-1\rangle. \quad (33)$$

Let us now consider two cases: (a) $C = 0$, (b) $C \neq 0$.

(a) $C = 0$. The formulas (32) and (33) in this case give

$$R(T_2)|j\rangle = -(s^{-1}q^{-j+1/2} + sq^{j+1/2})^{-1}|j+1\rangle, \quad (34)$$

$$iR(T_1)|j\rangle = \frac{sq^{j+1/2}}{s^{-1}q^{-j+1/2} + sq^{j+1/2}}|j+1\rangle. \quad (35)$$

If the set $\{|j\rangle \mid j = 0, 1, 2, \dots\}$ is linear independent, it follows from (34) and (35) that $\text{lin}\{|j\rangle, |j+1\rangle, \dots\}$ is invariant subspace for any $j = 1, 2, 3, \dots$. Thus the representation is either reducible or one-dimensional.

Now let there exist $l \in \mathbf{N}$ such that $|l\rangle$ is linear dependent on linear independent vectors $|0\rangle, |1\rangle, \dots, |l-1\rangle$. Since the sequence of numbers $[j]_{q,s}, j \in \mathbf{Z}$, does not contain 3 equal elements, the only possible case is $|l\rangle = \alpha|k\rangle$ for some $k \in \{0, 1, \dots, l-1\}$.

Let us consider the case $-sq^{l-1} \neq \pm i$. For $\alpha \neq 0$ we get contradiction with the commutation relations (1)–(3) applying them to the vector $|l-1\rangle$. For $\alpha = 0$ and $l \geq 2$ we get the one-dimensional representation on the invariant subspace $\mathbf{C}|l-1\rangle$. For $\alpha = 0$ and $l = 1$ we must move attention to the vectors $|0\rangle, |l-1\rangle, \dots$ and the rest of the proof is fulfilled by repeating the above and below arguments except that we work with vectors with negative indices.

Now let us consider the case $-sq^{l-1} = \varepsilon i$, $\varepsilon = \pm 1$. Since $sq^{l-1} = -s^{-1}q^{-l+1}$, the equalities (34) and (35) have no sense in this case.

For $\alpha \neq 0$ we get from equation $[l-1+j]_{q,s} = [l-1-j]_{q,s}$ (valid for all $j \in \mathbf{Z}$) that $k = l-2$. By (27), (30) and the relation $sq^{l-1} = -s^{-1}q^{-l+1}$ we have

$$|l\rangle = R(iT_1 - s^{-1}q^{-l+3/2}T_2)|l-1\rangle = \alpha|k\rangle = \alpha R(iT_1 + sq^{l-1/2}T_2)|l-1\rangle = 0.$$

This contradict the equality $|l\rangle = \alpha|k\rangle$, $\alpha \neq 0$. For $\alpha = 0$ we have from (1) and from $R(iT_1 - \varepsilon iq^{1/2}T_2)|l-1\rangle = 0$ that

$$R(I)R(T_2)|l-1\rangle = \varepsilon \frac{q^2 + 1}{q^2 - 1} R(T_2)|l-1\rangle = i[l]_{q,s} R(T_2)|l-1\rangle$$

and we can redefine $|l\rangle := R(T_2)|l-1\rangle$. From (3) and (30) we have $0 = R(iT_1 + sq^{l+1/2}T_2)|l\rangle$.

If $|l\rangle$ is linear dependent on $|0\rangle, |1\rangle, \dots, |l-1\rangle$, there exists $\beta \in \mathbf{C}$ such that $|l\rangle = \beta|l-2\rangle$. As above, for $\beta \neq 0$ we get the contradiction $\beta|l-1\rangle = 0$ and for $\beta = 0$ we get a one-dimensional representation since $\mathbf{C}|l-1\rangle$ is an invariant subspace.

If $|l\rangle$ is linear independent on the vectors $|0\rangle, |1\rangle, \dots, |l-1\rangle$, we recursively re-define $|j+1\rangle := R(iT_1 s^{-1} q^{-j+1/2} T_2)|j\rangle$, $j = l, l+1, \dots$. Then we again consider 2 cases:

If there exists $\gamma \in \mathbf{C}$ such that $|l+l'\rangle = \gamma|l-2-l'\rangle$ for some $l' \in \{1, 2, \dots, l-2\}$ then we get either one-dimensional representation on the invariant subspace $\mathbf{C}|l+l'-1\rangle$ (when $\gamma = 0$) or a contradiction applying (1)–(3) to $|l-2-l'\rangle$.

If $|l+l'\rangle$ is linear independent on $|0\rangle, |1\rangle, \dots, |l+l'-1\rangle$, then the representation is reducible since $\text{lin}\{|j\rangle, |j+1\rangle, \dots\}$ is invariant subspace for any j .

(b) $C \neq 0$. Consider first the case when $|l\rangle$ is linear dependent on the linear independent vectors $|0\rangle, |1\rangle, \dots, |l-1\rangle$. It means that $|l\rangle = \alpha|k\rangle$ for some $k \in \{0, 1, \dots, l-1\}$ and for $\alpha \in \mathbf{C}$. If $\alpha = 0$ we get a contradiction since

$$0 = |l\rangle = R(iT_1 - s^{-1}q^{-l+3/2}T_2)|l-1\rangle = R(iT_1 + sq^{l+1/2}T_2)|l\rangle = -Cq|l-1\rangle,$$

implies $C = 0$. For $\alpha \neq 0$ we get a contradiction applying (1)–(3) to the vector $|l-1\rangle$.

Now consider the case when the vectors $|j\rangle$, $j = 0, 1, 2, \dots$, are linear independent. If there exists $m \in \mathbf{N}$ such that the vector $|-m\rangle$ is linear dependent on the linear independent vectors $|j\rangle$, $j \in \{-m+1, -m+2, \dots\}$, we write $|-m\rangle = \beta|p\rangle$ for some $\beta \in \mathbf{C}$ and $p > -m$. For $\beta = 0$ we get a contradiction similarly as in the above analogous cases. For $\beta = Cq$ and $p = -m+1$ we get the representation given (after some suitable rescaling of the basis) by (22)–(26). For $\beta = -Cq$ and $p = -m+2$ we can derive from (1) how operator $R(I)$ acts on the linear independent vectors $|p\rangle$ and $R(T_2)|p-1\rangle$ and see that it cannot be diagonalized on this subspace. Therefore, this case is impossible. For other values of β and p we get a contradiction applying (1)–(3) to $|-m+1\rangle$.

Thus the only possible remaining case is when all the vectors $|j\rangle$, $j \in \mathbf{Z}$ are linear independent. Using the formulas (29), (32) and (33), in this case we get (after some suitable rescaling of the basis) the representation (18)–(20). Theorem is proved.

It is clear from Theorem 2 that for $q \in \mathbf{R}$ the irreducible $*$ -representations of $U_q(\text{iso}_2)$ which can be separated from representations of Theorem 1, are equivalent to the irreducible $*$ -representations from [7]. However, it is not seen directly from formulas for representations since operators of representations in [7] are given with respect to another basis than our one. Namely, the authors of [7] diagonalize the operator $R(T_2)$ which corresponds to the shifts in the group $ISO(2)$.

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